

Chapter 11 Parent Guide

Discrete Mathematics: Series and Patterns

A series is the sum of the terms of a sequence, and a sequence is an ordered list of numbers. Chapter 11 is a study of various patterns in numbers. The patterns will be arithmetic, geometric, or neither.

Patterns are sought in almost every walk of life, from wallpaper to economics. People gain confidence in their work when they can identify and act on certain patterns. Many students will find the topics in this chapter both challenging and fun.

Chapter 11 begins with defining terms used to describe sequences and series. Then arithmetic and geometric sequences and series are studied. Patterns in Pascal's triangle are explored in one lesson. The chapter ends with the binomial theorem used to expand (multiply) binomials raised to the n th degree.

Lesson 11.1 includes definitions used throughout the chapter and introduces the sigma notation used for summations of series. Lesson 11.2 discusses arithmetic sequences. Lesson 11.3 discusses arithmetic series. Lesson 11.4 discusses geometric sequences. Lesson 11.5 discusses geometric series and mathematical induction to prove statements about natural numbers. Lesson 11.6 involves infinite geometric series. Lesson 11.7 explores combinations and probabilities and other sequences found in Pascal's triangle. Lesson 11.8 introduces the binomial theorem.

You can help your child with the terminology by doing the following activity together. You may want to first review the definition of sequence at the top of page 691 and the discussion of arithmetic sequences on page 699.

PROBLEM FOR DISCUSSION (See textbook page 713)

An automobile that cost \$12,500 depreciates, and its value at the end of a given year is 80% of its value at the end of the preceding year. What is it worth after 10 years? You can answer this question by using a geometric sequence, which uses a common ratio for each of its successive terms.

1. Discuss how a geometric sequence fits the definition of sequence.

A sequence is an ordered list of numbers.

A geometric sequence is a sequence in which the ratio of successive terms is the same number, r , called the common ratio.

A geometric sequence is an ordered list of numbers that has a common ratio. This is just a specific example of a sequence.

2. Read the discussion on arithmetic sequences on page 699. Discuss how an arithmetic sequence differs from a geometric sequence.

The difference between an arithmetic sequence and a geometric sequence is that the terms in an arithmetic sequence have a common

difference and the terms in a geometric sequence have a common ratio.

3. Examine the graph for an arithmetic sequence on page 699. Compare it to the graph of a geometric sequence on page 713. How are the graphs alike? How are they different?

The graphs are alike in that they are both either increasing or decreasing.

The graphs are different in that the geometric graph increases or decreases at a faster rate than the arithmetic graph.

4. Suppose 4 and 12 are the first two terms of a geometric sequence. Discuss how to determine the common ratio. Then discuss how to determine the next term of the sequence that follows 12. How would your response differ if 4 and 12 were the first two terms of an arithmetic sequence?

To find the common ratio, r , divide 12 by 4: $\frac{12}{4} = \frac{3}{1}$.

To find the next term, multiply the ratio by 12. So $12 \times 3 = 36$.

If 4 and 12 were the first two terms of an arithmetic sequence, the common difference would be $12 - 4 = 8$. So the next term would be $12 + 8 = 20$.

The following are complete worked out solutions to selected exercises in the student textbook. These solutions are provided to you so that you can help your child with their homework. Your child's classroom notes, example problems in the text, and these worked out solutions are all useful tools to help you and your child work through their assignment.

Lesson 11.1

15. $t_n = 5n - 1$

$t_1 = 5(1) - 1 = 4$

$t_2 = 5(2) - 1 = 9$

$t_3 = 5(3) - 1 = 14$

$t_4 = 5(4) - 1 = 19$

4, 9, 14, 19

29. $t_1 = -6$

$t_n = -2t_{n-1} + 3$

$t_2 = -2t_1 + 3 = -2(-6) + 3 = 15$

$t_3 = -2t_2 + 3 = -2(15) + 3 = -27$

$t_4 = -2t_3 + 3 = -2(-27) + 3 = 57$

$t_5 = -2t_4 + 3 = -2(57) + 3 = -111$

$t_6 = -2t_5 + 3 = -2(-111) + 3 = 225$

-6, 15, -27, 57, -111, 225

$$\begin{aligned}
 46. \sum_{k=1}^5 -2k &= -2(1) - 2(2) - 2(3) - 2(4) - 2(5) \\
 &= -2 - 4 - 6 - 8 - 10 \\
 &= -30
 \end{aligned}$$

$$\begin{aligned}
 64. \sum_{m=1}^5 (2m^2 + 3m + 2) &= 2\sum_{m=1}^5 m^2 + 3\sum_{m=1}^5 m + \sum_{m=1}^5 2 \\
 &= 2\left[\frac{5(5+1)(10+1)}{6}\right] + 3\left[\frac{5(5+1)}{2}\right] + 5(2) \\
 &= 2(55) + 3(15) + 10 \\
 &= 165
 \end{aligned}$$

78. $\frac{180(n-2)}{4} = \frac{180(5-2)}{5} = 108^\circ$

Lesson 11.2

11. Yes, it is an arithmetic sequence and $d = 3$.

35. $t_1 = -4$

$t_2 = t_1 + 3 = -4 + 3 = -1$

$t_3 = t_2 + 3 = -1 + 3 = 2$

$t_4 = t_3 + 3 = 2 + 3 = 5$

-4, -1, 2, 5

45. -3, 2, 7, 12

Chapter 11

54. Step 1. Find d

$$\begin{aligned}t_6 &= t_1 + 5d \\ -2.85 &= 2.1 + 5d \\ d &= -0.99\end{aligned}$$

Step 2. $t_1 = 2.1$

$$\begin{aligned}\text{Step 3. } t_n &= t_1 + (n-1)d \\ t_{10} &= 2.1 + 9(-0.99) \\ t_{10} &= -6.81\end{aligned}$$

58. $d = 15 - 11 = 4$ and $t_1 = 11$
 $t_n = 11 + (n-1)(4)$

71. $t_1 = 18, t_5 = -10$

Step 1. Find d

$$\begin{aligned}t_5 &= t_1 + 4d \\ -10 &= 18 + 4d \\ d &= -7\end{aligned}$$

Step 2. $18 - 7 = 11$

$$11 - 7 = 4$$

$$4 - 7 = -3$$

The three arithmetic means are 11, 4, and -3 .

Lesson 11.3

11. $-100 + (-96) + (-92) + (-88)$

$$t_1 = -100, t_n = -88, n = 4$$

$$S_5 = 4\left(\frac{-100 - 88}{2}\right) = -376$$

19. $3 + 6 + 9 + \dots + 99$

$$t_1 = 3, t_n = 99, n = 33$$

$$S_n = n\left(\frac{t_1 + t_n}{2}\right)$$

$$S_{33} = 33\left(\frac{3 + 99}{2}\right) = 1683$$

31. $-10 + (-15) + (-20) + (-25) + (-30) + \dots$

$$t_1 = -10, d = -5, n = 25$$

$$S_n = n\left[\frac{2t_1 + (n-1)d}{2}\right]$$

$$S_{25} = 25\left[\frac{2(-10) + 24(-5)}{2}\right]$$

$$S_{25} = -1750$$

40. $\sum_{n=1}^5 (100 - 5n)$

$$t_1 = 100 - 5 = 95, d = -5, N = 5$$

$$S_N = N\left[\frac{2t_1 + (N-1)d}{2}\right]$$

$$S_5 = 5\left[\frac{2(95) + 4(-5)}{2}\right]$$

$$S_5 = 425$$

Lesson 11.4

13. Yes,
- $r = 5$

The next three terms are 1250, 6250 and 31,250.

- 32.
- $t_1 = -3$

$$t_2 = -2.2t_1 = -2.2(-3) = 6.6$$

$$t_3 = -2.2t_2 = -2.2(6.6) = -14.52$$

$$t_4 = -2.2t_3 = -2.2(-14.52) = 31.944$$

$$-3, 6.6, -14.52, 31.944$$

- 44.
- $t_7 = 10,935; t_{11} = 135$

Step 1. Find r .

$$\frac{t_{11}}{t_7} = \frac{135}{10,935}$$

$$\frac{t_1 r^{10}}{t_1 r^6} = \frac{1}{81}$$

$$r^4 = \frac{1}{81}$$

$$r = \frac{1}{3}$$

Step 2. Find t_1 .

$$t_7 = t_1 r^6$$

$$10,935 = t_1 \left(\frac{1}{3}\right)^6$$

$$t_1 = 7,971,615$$

Step 3. Find t_6 .

$$t_6 = t_1 r^5$$

$$= 7,971,615 \left(\frac{1}{3}\right)^5$$

$$= 32,805$$

The sixth term is 32,805.

- 51.
- $t_1 = \frac{1}{2} = \frac{1}{2}$

$$t_n = \left(\frac{1}{2}\right)^{n-1}$$

- 60.
- $t_1 = 1, t_5 = 81$

Step 1. Find r .

$$t_5 = t_1 r^4$$

$$81 = 1 \cdot r^4$$

$$r = \pm 3$$

Step 2. Find the means using each value of r .

$$r = 3: t_2 = t_1 r = 1(3) = 3$$

$$t_3 = t_2 r = 3(3) = 9$$

$$t_4 = t_3 r = 9(3) = 27$$

$$r = -3: t_2 = t_1 r = 1(-3) = -3$$

$$t_3 = t_2 r = -3(-3) = 9$$

$$t_4 = t_3 r = 9(-3) = -27$$

The geometric means are 3, 9, 27 or $-3, 9, -27$.

Chapter 11

76. $t_n = t_1 r^{n-1}$

After 4 years, $n = 5$.

$$t_5 = 1,200,000(1.05)^4$$

$$t_5 = 1,458,607.5$$

The value after 4 years will be \$1,458,607.50.

Lesson 11.5

14. $S_6 = 2\left(\frac{1 - (-3)^6}{1 - (-3)}\right) = 2\left(\frac{1 - 729}{1 + 3}\right) = -364$

26. $t_1 = 3, r = \frac{9}{3} = 3, n = 5$

$$S_n = t_1 \left(\frac{1 - r^n}{1 - r}\right)$$

$$S_5 = 3\left(\frac{1 - 3^5}{1 - 3}\right)$$

$$S_5 = 363$$

38. $\sum_{k=1}^{12} (3 \cdot 2^k) = 6 + 12 + \dots + 3 \cdot 2^{12}$

$$t_1 = 6, r = \frac{12}{6} = 2, t_{12} = 3 \cdot 2^{11} = 6144$$

$$S_{12} = 6\left(\frac{1 - 2^{12}}{1 - 2}\right)$$

$$S_{12} = 24,570$$

47. $\sum_{m=1}^7 3(0.2^{m-1})$

$$t_1 = 3, r = 0.2, n = 7$$

$$S_7 = 3\left(\frac{1 - 0.2^7}{1 - 0.2}\right)$$

$$S_7 \approx 3.8$$

60. $2 \leq n + 1$

1. Basis Step

Show that $2 \leq n + 1$ is true for $n = 1$.

$$2 \leq 1 + 1 \text{ True}$$

2. Induction Step

Assume the statement is true for a natural number k .

$$S_k: 2 \leq k + 1$$

Determine the statement to be proved:

$$S_{k+1}: 2 + 1 \leq (k + 1) + 1$$

$$3 \leq k + 2$$

But if $2 \leq k + 1$, by the Addition Property of Inequality:

$$2 + 1 \leq (k + 1) + 1$$

$$3 \leq k + 2 \text{ True}$$

Chapter 11

67. $t_1 = 1600, r = \frac{1}{4}$

$$S_6 = 1600 \left(\frac{1 - \left(\frac{1}{4}\right)^6}{1 - \frac{1}{4}} \right)$$

$$S_6 = 2132.8125$$

The sum of the areas of the first six squares is 2132.8125 in^2 .

Lesson 11.6

9. $2 + 1.5 + 1.125 + 0 + 0.84375 + \dots, t_1 = 2, r = 0.75$

$$\begin{aligned} S &= \frac{t_1}{1-r} \\ &= \frac{2}{1-0.75} \\ &= 8 \end{aligned}$$

21. $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k, t_1 = \frac{1}{8}, r = \frac{1}{8}$

$$\begin{aligned} S &= \frac{t_1}{1-r} \\ &= \frac{\frac{1}{8}}{1-\frac{1}{8}} \\ &= \frac{1}{7} \end{aligned}$$

43. $0.353535\dots$

$$= 0.35 + 0.0035 + 0.000035 + \dots$$

$$= \frac{35}{100} + \frac{35}{10,000} + \frac{35}{1,000,000} + \dots$$

$$= \sum_{k=1}^{\infty} 35 \left(\frac{1}{100}\right)^k$$

52. $0.\overline{43} = 0.434343\dots$

$$= 0.43 + 0.0043 + 0.000043 + \dots$$

$$= \frac{43}{100} + \frac{43}{10,000} + \frac{43}{1,000,000} + \dots$$

$$t_1 = \frac{43}{100}, r = \frac{1}{100}$$

$$\begin{aligned} S &= \frac{t_1}{1-r} \\ &= \frac{\frac{43}{100}}{1-\frac{1}{100}} \\ &= \frac{43}{99} \end{aligned}$$

Lesson 11.7

9. ${}_8C_5$ is the sixth entry in the eighth row, ${}_8C_5 = 56$

30. $P(\text{at least 3 heads}) = P(\text{exactly 3 heads}) + P(\text{exactly 4 heads}) + P(\text{exactly 5 heads})$

$$\begin{aligned} &= \frac{{}_5C_3}{2^5} + \frac{{}_5C_4}{2^5} + \frac{{}_5C_5}{2^5} \\ &= \frac{10 + 5 + 1}{2^5} \\ &= 0.5 \end{aligned}$$

37. $P(\text{at least 4 boys}) = P(\text{exactly 4 boys}) + P(\text{exactly 5 boys})$

$$\begin{aligned} &= \frac{{}_5C_4}{2^5} + \frac{{}_5C_5}{2^5} \\ &= \frac{5 + 1}{32} \\ &\approx 0.19 \end{aligned}$$

42. $P(\text{exactly 4}) = \frac{{}_5C_4}{2^5}$

$$\begin{aligned} &= \frac{5}{32} \\ &\approx 0.16 \end{aligned}$$

Lesson 11.8

10. $(p + q)^6 = \binom{6}{0}p^6q^0 + \binom{6}{1}p^5q^1 + \binom{6}{2}p^4q^2 + \binom{6}{3}p^3q^3 + \binom{6}{4}p^2q^4 + \binom{6}{5}p^1q^5 + \binom{6}{6}p^0q^6$

$$= p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

23. $\sum_{k=0}^5 \binom{5}{k} a^{5-k} b^k = (a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

28. fifth term, $k = 4$: $\binom{9}{4} r^5 s^4 = 126r^5 s^4$

41. $(x - 2y)^4 = \binom{4}{0}x^4(-2y)^0 + \binom{4}{1}x^3(-2y)^1 + \binom{4}{2}x^2(-2y)^2 + \binom{4}{3}x^1(-2y)^3 + \binom{4}{4}(x)^0(-2y)^4$

$$= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$$

52. $n = 18$, so there are $n + 1$, or 19 terms in the expansion

60. $P(\text{exactly 3 hits}) = \binom{6}{3}(0.285)^3(0.715)^3$

$$\approx 0.169$$

The probability is about 0.17.